PLASTIC FLOW FOR AXISYMMETRIC-TENSION SPECIMENS WITH NECKING

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Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 10, No. 2, pp. 141–144, 1969

Plastic deformation of a tension specimen bounded by a curved surface of revolution is considered. Such a configuration may occur, for example, as a result of necking. The specimen material satisfies the Tresca yield condition and the associated flow rule. Approximate solutions for the stress distribution in the neck were examined in [6]. The extension of notched bars was investigated by numerical and graphic methods in [2, 4]. Below, the problem is solved analytically for a sufficiently smooth neck; a small degree of nonuniformity of the plastic properties is taken into account.

1. We assume that the plastic state in question corresponds to the Haar-Karman regime [1]; as usual, the validity of this assumption is confirmed by the possibility of obtaining a complete solution. In tension the radial component of the flow rate $u < 0$ and, in accordance with the regime adopted, the circumferential principal stress are given by

$$\sigma_\theta = {}^1/_2\ (\sigma_r + \sigma_z) - k$$

while the components σ_r, σ_z, τ_{rz} must satisfy the equilibrium and yield conditions

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_z}{2r} + \frac{k}{r} = 0, \qquad \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0$$
$$(\sigma_r - \sigma_z)^2 + 4\tau_{rz}^2 = 4k^2. \tag{1.1}$$

Here, k is the yield point of the material in shear. Plastic flow occurs in the region AOB (Fig. 1) adjacent to the minimum cross section of the test piece. The absence of stresses at the free surface makes it possible to write the boundary conditions for system (1.1):

$$\sigma_r = k - k\cos 2\lambda, \quad \sigma_z = k + k\cos 2\lambda, \ \tau_{rz} = k \sin 2\lambda \tag{1.2}$$

where λ is the angle between the tangent to AB and the oz-axis.

The shape of the neck outside AB does not affect the solution; however, it must be such that the yield condition is nowhere exceeded. On AB we assume that the neck is formed by a smooth curve whose equation can be written in the form

$$\frac{r}{a} = 1 + \delta\varphi\left(\frac{z}{a}\right), \qquad \varphi(0) = \varphi'(0) = 0 \tag{1.3}$$

and the distribution of the mechanical properties in the plastic region

$$k = k(r) + \varepsilon K(r, z). \tag{1.4}$$

Here, a is the radius of the minimum cross section and δ and ε are small dimensionless parameters of the same order.

Plastic nonuniformity of the (1.4) type may occur, for example, as a result of hardening during extension.

In Eqs. (1.1)–(1.4) it is possible to transform to dimensionless quantities and in what follows all the geometric dimensions will be referred to the characteristic dimension a, and the stresses to k(r).

We write the linearized solution of system (1.1) in the parameters δ and ε:

$$\sigma_{ij} = \sigma_{ij}^\circ + (\delta + \varepsilon)\,\sigma_{ij}'. \tag{1.5}$$

The case $\delta = \varepsilon = 0$ corresponds to the extension of a circular cylinder, when the yield stress is a function only of the coordinate r, and for the zero-order solution we have

$$\sigma_z^\circ = 2, \qquad \sigma_r^\circ = \tau_{rz}^\circ = 0. \tag{1.6}$$

Substituting (1.5), (1.6) into (1.1) and linearizing, we obtain

$$\frac{\partial \sigma_r'}{\partial r} + \frac{\partial \tau_{rz}'}{\partial z} = 0, \quad \frac{\partial \tau_{rz}'}{\partial r} + \frac{\partial \sigma_r'}{\partial z} + m\frac{\partial K}{\partial z} + \frac{\tau_{rz}'}{r} = 0$$
$$\sigma_z' = \sigma_r' + mK \tag{1.7}$$

and, linearizing (1.2), (1.3),

$$\sigma_r' = 0, \quad \sigma_z' = mK, \quad \tau_{rz}' = (2-m)\frac{d\varphi(z)}{dz} \quad \text{at } r = 1, \quad \left(m = 2\frac{\varepsilon}{\delta + \varepsilon}\right). \tag{1.8}$$

The solution of Eqs. (1.7) with boundary conditions (1.8) (Cauchy problem) will be found in series form:

$$\sigma_r' = N_0(z) + \sum_{i=1}^{\infty} N_i(z)(1-r)^i$$
$$\sigma_z' = S_0(z) + \sum_{i=1}^{\infty} S_i(z)(1-r)^i$$
$$\tau_{rz}' = T_0(z) + \sum_{i=1}^{\infty} T_i(z)(1-r)^i, \tag{1.9}$$

for which purpose we also expand the function K(r, z) in powers of (1−r):

$$K(r, z) = K_0(z) + \sum_{i=1}^{\infty} K_i(z)(1-r)^i. \tag{1.10}$$

The convergence of series (1.9) and (1.10) is ensured by the choice of $\varphi(z)$ and K(r, z) as analytic functions of their arguments [3].

Satisfying boundary conditions (1.8), we have

$$N_0 = 0, \quad S_0 = mK_0, \quad T_0 = (2-m)\frac{d\varphi(z)}{dz}. \tag{1.11}$$

Substituting (1.9) and (1.10) into Eqs. (1.7) and equating to zero, terms containing the same powers of (1−r), we find recurrence relations for the successive computation of the remaining coefficients of series (1.9),

$$N_i = \frac{1}{i} T'_{i-1}, \quad S_i = N_i + mK_i, \quad i = 1, 2, 3, \ldots$$
$$T_1 = N_0' + T_0 + mK_0'$$
$$T_i = \frac{1}{i}(iT_{i-1} - N'_{i-2} + N'_{i-1} - mK'_{i-2} + mK'_{i-1}), \quad i = 2, 3, 4, \ldots. \tag{1.12}$$

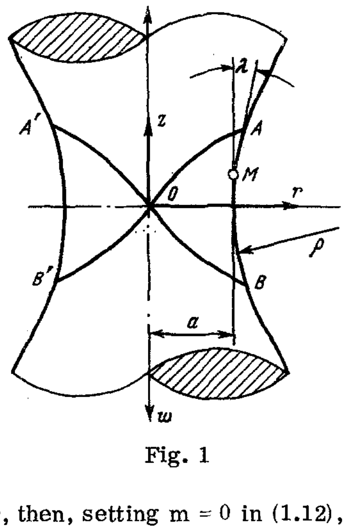

Fig. 1

If the nonuniformity depends only on r, then, setting m = 0 in (1.12), we obtain

$$N_1 = T_0', \quad T_1 = T_0, \quad N_2 = {}^1/_2\, T_0, \quad T_2 = T_0 + {}^1/_2\, T_0''$$
$$N_3 = {}^1/_3 T_0 + {}^1/_6\, T_0''', \quad T_3 = T_0 + {}^1/_3 T_0'' \tag{1.13}$$

$$N_4 = {}^1/_4 T_0' + {}^1/_{12} T_0''', \quad T_4 = T_0 + {}^7/_{24} T_0'' + {}^1/_{24} T_0''''; \ldots$$
$$S_i = N_i, \quad i = 1, 2, 3, \ldots . \tag{1.14}$$

In (1.12) and (1.13) the primes denote derivatives with respect to z.

Near the axis of symmetry the terms in Eqs. (1.1) containing r^{-1} vanish [4], and the stress field can be continued, following [3].

2. Consider the velocity field. The components of u and w in the directions or, oz must satisfy the incompressibility condition

$$\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} + \frac{u}{r} = 0 \tag{2.1}$$

and the isotropy condition

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = 0. \tag{2.2}$$

The boundary conditions are given on the slip lines OA and OB, along which the condition of continuity of the normal component must be satisfied. In accordance with (1.5) we set

$$u = u^\circ + (\delta + \varepsilon)\, u', \quad w = w^\circ + (\delta + \varepsilon)\, w'.$$

If the rigid parts of the test piece are displaced at rates w = V, w = −V, then for the zero-order solution the velocity field will be [1]

$$\frac{u^\circ}{V} = -\frac{2}{\pi}\left[1 - \left(\frac{z}{r}\right)^2\right]^{1/2}, \quad \frac{w^\circ}{V} = 1 - \frac{2}{\pi} \arccos\left(\frac{z}{r}\right). \tag{2.3}$$

Linear equations (2.1) and (2.2) are retained for the quantities u' and w', and linearization of the boundary conditions along OA and OB gives

$$u' + w' = 0 \ \text{at} \ r = z, \ r = -z.$$

It is easy to see that these conditions are satisfied by the solution

$$u' = w' = 0.$$

Thus, correct to quantities of the second order, the shape of the neck (1.3) does not affect the velocity field (2.3) of the zero-order solution. This is also true of the case of plane deformation [5]; it continues to apply in the presence of nonuniformity of the (1.4) type.

3. In analyzing the state of stress in the neck of a tensile test piece, functions (1.3) and (1.4) must usually be determined experimentally. Solution (1.9) is especially simple if $\varphi(z)$ is an integral polynomial of degree n; in this case the derivatives of T_0, starting from the n-th, vanish and series (1.9) can be summed.

For simplicity, we consider the extension of a homogeneous rod with a paraboloidal neck (Fig. 1):

$$r = 1 + \delta z^2. \tag{3.1}$$

The coefficients (1.13) will be

$$N_0 = 0, \quad N_i = \frac{1}{i} T_0' = \frac{4}{i}, \quad i = 1, 2, 3, \ldots$$
$$T_0 = T_i = 4z$$

and summation gives

$$\sigma_r' = \sigma_z' = -4 \ln r, \quad \tau_{rz}' = 4z/r. \tag{3.2}$$

Near the axis of symmetry ($r \leq \gamma$, $\gamma \to 0$) the solution becomes

$$\sigma_r' = \sigma_z' = A - 4\frac{r}{\gamma}, \quad \tau_{rz}' = 4\frac{z}{\gamma}. \tag{3.3}$$

Constant A is determined from (3.2) and (3.3):

$$A = 4(1 - \ln \gamma).$$

The boundary of the region of solution (3.3) is now known; however, the finiteness of the stresses on the axis requires that γ be of the same order as δ. Correct to quantities of order δ^2, the mean yield stress $\langle \sigma_z \rangle$ can be found by integrating (1.6) and (3.2):

$$\langle\sigma_z\rangle = \frac{1}{\pi}\int_0^1 2\pi r\,(2-4\delta\ln r)\,dr = 2(1+\delta) = 2\left(1+\frac{1}{2\rho}\right) \tag{3.4}$$

where ρ is the radius of curvature of the meridional section of the paraboloidal neck at z = 0.

We note that Eq. (3.4) differs from the corresponding solution of Davidenko and Spiridonova [6] with respect to the coefficient of the term $1/\rho$.

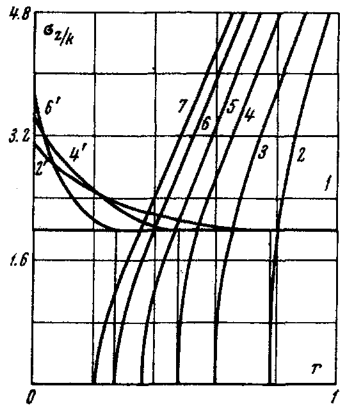

Fig. 2

We investigate the development of the neck and the associated stress distribution in the state preceding tensile fracture using the following model. After the yield point is reached, plastic flow develops in the neighborhood of the weakest section; as a result of hardening the location of this section changes continuously and plastic deformation successively embraces different parts of the test piece, hardening it uniformly (in the statistical sense) and preserving the initial cylindrical shape. However, during deformation the metal loses its ability to harden and after a certain time the flow region is localized, causing the formation of a neck. An experimental confirmation of this model may be found in [7]. As shown, in the first approximation, for an arbitrary neck described by an equation of the (1.3) type the velocity field (2.3) is preserved. From Eqs. (2.3) we computed the successive changes in the neck shown in Fig. 2 (curves 1, . . ., 7). In each stage of the computations the shape of the neck was approximated by Eq. (3.1) and the smallness of the parameter δ was checked. The computations were continued until the starting diameter was reduced by a factor of 4.8; however, the value of δ did not exceed 0.15.

In Fig. 2 we have plotted the vertical stress diagrams (curves 2', 4', 6') in the minimum cross section of the test piece calculated from Eqs. (3.2) and (3.3) for the instants in question.

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